AD 46099

Technical Report No. 5

RESPONSE OF AN ELASTIC DISK TO IMPACT AND MOVING LOADS

By

A. Cemal Eringen

to

Office of Naval Research
Department of the Navy

Contract Nonr-1100(02)

Division of Engineering Sciences
Purdue University
Lafayette, Indiana

1 September 1954

DISTRIBUTION LIST

1. Administrative, Reference and Liaison Activities of ONR

Chief of Naval Research
Department of the Navy
Washington 25, D.C.
Attn: Code 438 (2)
Code 432 (1)
Code 466 (via Code 108) (1)

Director, Naval Research Lab.
Washington 25, D.C.
Attn: Tech. Info. Officer (9)
Technical Library (1)
Mechanics Division (2)

Library of Congress
Washington 25, D.C.
Attn: Navy Research Section (2)

Office of Naval Research
Branch Office
The John Crerar Library Building
Tenth Floor, 86 E. Randolph St.
Chicago 1, Illinois (2)

II. Department of Defense and other interested Government activities

(A) GENERAL

Research and Development Board
Department of Defense
Pentagon Bldg.
Washington 25, D.C.
Attn: Library (Code 3D-1075) (1)

Armed Forces Special Weapons Project
P.O. Box 2610
Washington, D.C.
Attn: Col. G. F. Blunda (2)

Joint Task Force 3
12 St. and Const. Ave., N.W. (Temp. U)
Washington 25, D.C.
Attn: Major B. D. Jones (1)

(B) ARMY

Chief of Staff
Department of the Army
Research and Development Div.
Washington 25, D.C.
Attn: Chief of Res. and Dev. (1)

Office of the Chief of Engineers
Assistant Chief for Works
Department of the Army
Bldg. T-7, Gravelly Point
Washington 25, D.C.
Attn: Structural Branch (R. L. Bloor) (1)

Engineering Research and Development Laboratory
Fort Belvoir, Virginia
Attn: Structures Branch (1)

Office of the Chief of Engineers
Asst. Chief for Military Construction
Department of the Army
Bldg. T-3, Gravelly Point
Washington 25, D.C.
Attn: Structures Branch (M.F.Carey) (1)
Protective Construction Branch (I.O.Thorley) (1)

Office of the Chief of Engineers
Asst. Chief for Military Operations
Department of the Army
Bldg. T-7, Gravelly Point,
Washington 25, D.C.
Attn: Structures Development Branch (W.F.Woollard) (1)

U.S. Army Waterways Experiment Station
P. O. Box 631
Halls Ferry Road
Vicksburg, Mississippi
Attn: Col. H. J. Skidmore (1)

The Commanding General
Sandia Base, P. O. Box 5100
Albuquerque, New Mexico
Attn: Col. Canterbury (1)

Operations Research Officer
Department of the Army
Ft. Lesley J. McNair
Washington 25, D.C. (1)
Attn: Howard Brackney

Office of Chief of Ordnance
Office of Ordnance Research
Department of the Army
The Pentagon Annex No. 2
Washington 25, D.C. (1)
Attn: ORDTB-PS

Ballistics Research Laboratory
Aberdeen Proving Ground
Aberdeen, Maryland
Attn: Dr. C. W. Lampson (1)

(C) Navy

Chief of Naval Operations
Department of the Navy
Washington 25, D.C.
Attn: OP-31 (1)
OP-363 (1)

Chief of Bureau of Ships
Department of the Navy
Washington 25, D.C.
Attn: Director of Research (2)
Code 423 (1)
Code 442 (1)
Code 421 (1)

Director, David Taylor Model Basin
Department of the Navy
Washington 7, D.C.
Attn: Code 720, Structures Division (1)
Code 740, Hi-Speed Dynamics Div., (1)

Commanding Officer,
Underwater Explosion Research Div.
Code 290
Norfolk Naval Shipyard
Portsmouth, Virginia (1)

Commander
Portsmouth Naval Shipyard
Portsmouth, N.H.
Attn: Design Division (1)

Director, Materials Laboratory
New York Naval Shipyard
Brooklyn 1, New York (1)

Chief of Bureau of Ordnance
Department of the Navy
Washington 25, D.C.
Attn: Ad-3, Technical Library (1)
Roc, P. H. Girouard (1)

Naval Ordnance Laboratory
White Oak, Maryland
RFD 1, Silver Spring, Maryland
Attn: Mechanics Division (1)
Explosive Division (1)
Mech. Evaluation Div. (1)

(C) NAVY

Commander
U.S.Naval Ordnance Test Station
Inyokern, California
Post Office- China Lake, Calif.
Attn: Scientific Officer (1)

Naval Ordnance Test Station
Underwater Ordnance Division
Pasadena, California
Attn: Structures Division (1)

Chief of the Bureau of Aeronautics
Department of the Navy
Washington 25, D.C.
Attn: TD-41, Technical Library (1)

Chief of Bureau of Ships
Department of the Navy
Washington 25, D.C.
Attn: Code P-314 (1)
Code C-313 (1)

Officer in Charge
Naval Civil Engr. Research and
Evaluation Laboratory
Naval Station
Port Hueneme, California (1)

Superintendent
U.S.Naval Post graduate School
Annapolis, Maryland (1)

(D) AIR FORCES

Commanding General
U. S. Air Forces
The Pentagon
Washington 25, D.C.
Attn. Res. & Dev. Div. (1)

Deputy Chief of Staff, Operations
Air Targets Division
Headquarters, U.S. Air Forces
Washington 25, D.C.
Attn: AFOIN-T/PV (1)

Flight Research Laboratory
Wright-Patterson Air Force Base
Dayton, Ohio
Attn: Chief, Applied Mechanics
Group (1)

(E) OTHER GOVERNMENT AGENCIES

U.S. Atomic Energy Commission
Division of Research
Washington, D.C. (1)

Director, National Bureau of
Standards
Washington, D.C.
Attn: Dr. W. H. Ramberg (1)

Forest Products Laboratory
Madison 5, Wisconsin (1)

Supplementary Distribution List

Professor Lynn Beedle
Fritz Engineering Laboratory
Lehigh University
Bethlehem, Pennsylvania (1)

Prof. R. L. Bisplinghoff
Dept. Aeronautical Engineering
Massachusetts Inst. of Tech.
Cambridge 39, Massachusetts (1)

Professor G. F. Carrier
Graduate Div. of Applied Mathematics
Brown University
Providence, Rhode Island (1)

Professor R. J. Dolan
Dept. of Theoretical and Applied
Mechanics
University of Illinois
Urbana, Illinois (1)

Professor Hans Bleich
Dept. of Civil Engineering
Columbia University
Broadway at 117th St.
New York 27, New York (1)

Prof. B. A. Boley
Dept. Aeronautical Engineering
Ohio State University
Columbus, Ohio (1)

Professor Lloyd Donnell
Department of Mechanics
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois (1)

Professor B. Fried
Dept. Mechanical Engineering
Washington State College
Pullman, Washington (1)

Mr. Martin Goland
Midwest Research Institute
4049 Pennsylvania Avenue
Kansas City 2, Missouri (1)

Dr. J. N. Goodier
School of Engineering
Stanford University
Stanford, California (1)

Professor M. Hetenyi
Walter P. Murphy Professor
Northwestern University
Evanston, Illinois (1)

Dr. N. J. Hoff, Head,
Dept. Aeronautical Engineering & Applied Mechanics
Polytechnic Institute of Brooklyn
99 Livingston Street
Brooklyn 2, New York (1)

Dr. J. H. Hollomon
General Electric Research Laboratories
1 River Road
Schenectady, New York (1)

Professor L. S. Jacobsen
Dept. of Mechanical Engineering
Stanford, California (1)

Dr. W. H. Hoppmann
Dept. of Applied Mechanics
Johns Hopkins University
Baltimore, Maryland (1)

Professor George Lee
Department of Mathematics
Renssalaer Polytechnic Institute
Troy, New York (1)

Professor Glen Murphy, Head
Dept. of Theoretical & Applied Mechanics
Iowa State College
Ames, Iowa (1)

Prof. N. M. Newmark
Dept. of Civil Engineering
University of Illinois
Urbana, Illinois (1)

Professor Jesse Ormondroyd
University of Michigan
Ann Arbor, Michigan (1)

Sandia Corporation
Sandia Base
Albuquerque, New Mexico
Attn: Mr. E. F. Cox
Code 5110 (1)

Dr. A. Phillips
School of Engineering
Stanford University
Stanford, California (1)

Dr. W. Prager, Chairman
Graduate Division of Applied Mathematics
Brown University
Providence 12, R. I. (1)

Dr. S. Raynor
Armour Research Foundation
Illinois Institute of Technology
Chicago 16, Illinois (1)

Professor E. Reissner
Dept. of Mathematics
Massachusetts Institute of Tech.
Cambridge 39, Massachusetts (1)

Professor M. A. Sadowsky
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois (1)

Professor Paul Lieber
Dept. of Aeronautical Engineering
Renssalaer Polytechnic Institute
Troy, New York (1)

Professor J. E. Stallmeyer
Talbot Laboratory
Dept. of Civil Engineering
University of Illinois
Urbana, Illinois (1)

Dr. C. B. Smith
College of Arts and Sciences
Department of Mathematics
Walker Hall
University of Florida
Gainesville, Florida (1)

Professor J. R. Andersen
Towne School of Engineering
University of Pennsylvania
Philadelphia, Pennsylvania (1)

Commander
U. S. Naval Proving Ground
Dahlgren, Virginia (1)

Professor E. Sternberg
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois (1)

Professor F. K. Teichmann
Dept. of Aeronautical Engineering
New York University
University Heights, Bronx
New York, N. Y. (1)

Professor C. T. Wang
Dept. of Aeronautical Engineering
New York University
University Heights, Bronx
New York, N. Y. (1)

Project File (2)

Project Staff (5)

For possible future distribution
by the University (10)

RESPONSE OF AN ELASTIC DISK TO IMPACT AND MOVING LOADS

by

A. Cemal Eringen *

Purdue University

ABSTRACT

With the use of Fourier transforms a class of elastodynamic problems concerning disks have been solved. The disk is subjected to various types of dynamic loadings at the rim. The case of impact and the moving loads are studied in detail.

(*) Associate Professor of Division of Engineering Sciences.

1. Introduction

Cylindrical roller bearings in high speed mechanism are subject to dynamic loadings. Yet the usual design procedure is based on Hertz formulae which are the result of elasto-static considerations. In many other instances, gears, rollers, or disks are subject to impact or moving loads. If we neglect the coriolis terms we can also bring the rolling disks on contact into the category of disks subject to moving loads. Thus the aim of the present paper is to obtain the solution to this class of elasto-dynamic problems concerning the disk. The dynamic load is applied to the rim of the disk. Two normal concentrated dynamic loads at the two ends of a diameter moving or otherwise are special cases.

Some solutions of free oscillation of cylinders are known since the time of Pochhammer [1], and later Pickett [2], J. Mindlin [3], T. Ghosh [4]. Similarly, the problem of rotating disks has attracted attention of many authors (see, for instance, Lamb and Southwell [5], Timoshenko and Goodier [6], Love [7]). It seems, however, that the forced oscillation problems concerning disks and cylinders have escaped the attention of authors, excepting a paper by J. Mindlin [8], which consists of generalities in a related problem to ours.

The present method is applicable to ring problems and to the plane elasto-dynamic problems concerning circular holes. These problems will be treated in later papers.

2. Formulation of the Problem

Equations of motion of plane homogeneous isotropic media in terms of plane polar coordinates r and θ and the time t are [7]: (See Fig. 1)

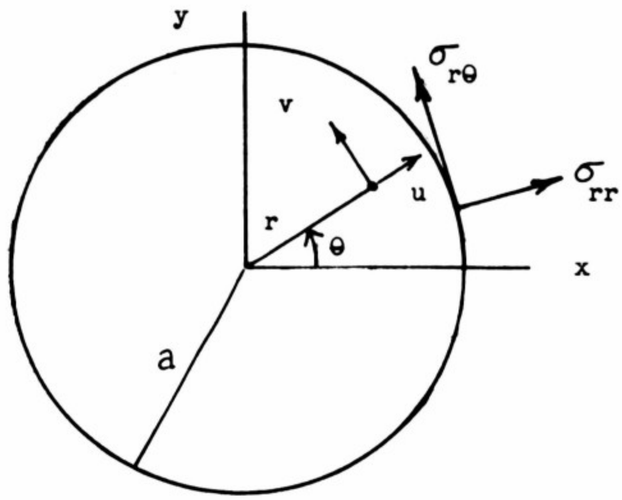

Fig. 1. Circular Disk

$$\gamma u_{,tt} = (\lambda + 2\mu)\Delta_{,r} - 2\mu r^{-1}\omega_{,\theta}$$
$$\gamma v_{,tt} = (\lambda + 2\mu) r^{-1}\Delta_{,\theta} + 2\mu\,\omega_{,r} \qquad (1)$$

where u (r, θ, t) and v (r, θ, t) are components of the displacement vector, λ and μ are the Lamé constants, and γ is the mass density per unit volume. Dilatation Δ and rotation ω are related to u, v by:

$$r\Delta = (r u)_{,r} + v_{,\theta} \quad , \quad 2 r \omega = (r v)_{,r} - u_{\theta} \qquad (2)$$

Subscripts after a comma represent differentiation, i.e., $u_{,\theta} = \partial u / \partial \theta$ etc. ....

Elimination of u and v among (1) and (2) leads to:

$$\alpha_1^2 r^2 \Delta_{,tt} = r(r\Delta_{,r})_{,r} + \Delta_{,\theta\theta}, \quad \alpha_1^2 = \gamma/(\lambda + 2\mu)$$
$$\alpha_2^2 r^2 \omega_{,tt} = r(r\omega_{,r})_{,r} + \omega_{,\theta\theta}, \quad \alpha_2^2 = \gamma/\mu \tag{3}$$

These are the equations of dilatational and rotational waves.

Components σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ of the stress tensor are given by:

$$\sigma_{rr} = \lambda\Delta + 2\mu u_{,r}, \quad \sigma_{r\theta} = \mu r^{-1} u_{,\theta} + \mu r(v/r)_{,r}$$
$$\sigma_{\theta\theta} = \lambda\Delta + 2\mu r^{-1}(v_{,\theta} + u) \tag{4}$$

The problem is to solve (1), (2) under a given σ_{rr} and $\sigma_{r\theta}$ at the rim $r = a$ of the disk:

$$\sigma_{rr}(a, \theta, t) = \sigma_o(\theta, t), \quad \sigma_{r\theta}(a, \theta, t) = \tau_o(\theta, t) \tag{5}$$

subject to the condition that these surface tractions are in equilibrium at each instant.

3. The Solution

The periodic solution of (3) with respect to θ is obtained to be

$$\bar{\Delta} = \sum_{n=o}^{\infty} (A_{1n} \sin n\theta + A_{2n} \cos n\theta)\, Z_n(\rho_1),$$
$$\bar{\omega} = \sum_{n=o}^{\infty} (B_{1n} \cos n\theta - B_{2n} \sin n\theta)\, Z_n(\rho_2), \tag{6}$$

$$Z_n(\rho_j) = C_{jn} J_n(\rho_j) + D_{jn} Y_n(\rho_j), \quad \rho_j = \alpha_j \tau r, \ (j = 1,2)$$

where $Z_n(\rho)$ is the cylinder function [9]; A_{jn}, B_{jn}, C_{jn} and D_{jn} are

constants of integration; and the barred quantities represent the Fourier transforms, i.e.

$$\bar{F}(\tau) = \int_{-\infty}^{\infty} e^{i\tau t} F(t)\, dt \quad , \quad i = \sqrt{-1} \tag{7}$$

The inversion formula for (7) is

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\tau} \bar{F}(\tau)\, d\tau \tag{8}$$

Substituting (6) into (2) after taking the Fourier transforms of (2) and solving the resulting partial differential equations we obtain:

$$\begin{aligned} \bar{u} &= \sum_{n=o}^{\infty} U_{1n}(r) \sin n\theta + U_{2n}(r) \cos n\theta \\ \bar{v} &= \sum_{n=o}^{\infty} V_{1n}(r) \cos n\theta - V_{2n}(r) \sin n\theta \end{aligned} \tag{9}$$

where

$$\begin{aligned} -r^{-1} U_{jn}(r) &= A_{jn} \rho_1^{-1} Z_n'(\rho_1) + B_{jn} 2n \rho_2^{-2} Z_n(\rho_2) \\ -r^{-1} V_{jn}(r) &= A_{jn} n \rho_1^{-2} Z_n(\rho_1) + B_{jn} 2\rho_2^{-1} Z_n'(\rho_2) \end{aligned} \tag{10}$$

where prime represents differentiation.

Combining (4), (6), and (9) we obtain

$$\begin{aligned} \bar{\sigma}_{rr}/2\mu &= \sum_{n=o}^{\infty} [A_{1n} N_{1n}(r\tau) + B_{1n} N_{2n}(r\tau)] \sin n\theta + [A_{2n} N_{1n}(r\tau) \\ &\qquad + B_{2n} N_{2n}(r\tau)] \cos n\theta \\ \bar{\sigma}_{r\theta}/2\mu &= \sum_{n=o}^{\infty} [A_{1n} S_{1n}(r\tau) + B_{1n} S_{2n}(r\tau)] \cos n\theta - [A_{2n} S_{1n}(r\tau) \\ &\qquad + B_{2n} S_{2n}(r\tau)] \sin n\theta \end{aligned} \tag{11}$$

(cont'd.)

(11, cont'd.)

$$\bar{\sigma}_{\theta\theta}/2\mu = \sum_{n=0}^{\infty} [A_{1n} T_{1n}(r\tau) + B_{1n} T_{2n}(r\tau)] \sin n\theta + [A_{2n} T_{1n}(r\tau) + B_{2n} T_{2n}(r\tau)] \cos n\theta$$

where:

$$N_{1n}(r\tau) = (\lambda/2\mu)\, Z_n(\rho_1) + (1 - n^2\rho_1^{-2})\, Z_n(\rho_1) + \rho_1^{-1} Z_n'(\rho_1)$$

$$N_{2n}(r\tau) = 2n\rho_2^{-2} Z_n(\rho_2) - 2n\rho_2^{-1} Z_n'(\rho_2)$$

$$S_{1n}(r\tau) = n\rho_1^{-2} Z_n(\rho_1) - n\rho_1^{-1} Z_n'(\rho_1) \tag{12}$$

$$S_{2n}(r\tau) = (1 - 2n^2\rho_2^{-2})\, Z_n(\rho_2) + 2\rho_2^{-1} Z_n'(\rho_2)$$

$$T_{1n}(r\tau) = (\lambda/2\mu)\, Z_n(\rho_1) + n^2\rho_1^{-2} Z_n(\rho_1) - \rho_1^{-1} Z_n'(\rho_1)$$

$$T_{2n}(r\tau) = -N_{2n}(r\tau)$$

4. Dynamic tractions applied to the rim of a disk

In the case of a disk, the stress and deformation components must be finite at $r = 0$. Hence $D_{jn} = 0$. Without loss of generality we also take $C_{jn} = 1$. This means in all of our formulas we must replace Z_n by J_n.

We use boundary conditions (5) to determine the constants A_{jn} and B_{jn}. The Fourier's theorem thus leads to:

$$A_{1n} = [2\mu D_n(a\tau)]^{-1} [S_{2n}(a\tau)\, \tilde{\sigma}_{os} - N_{2n}(a\tau)\, \tilde{\tau}_{oc}]$$

$$B_{1n} = [2\mu D_n(a\tau)]^{-1} [-S_{1n}(a\tau)\, \tilde{\sigma}_{os} + N_{1n}(a\tau)\, \tilde{\tau}_{oc}]$$

$$A_{2n} = [2\mu D_n(a\tau)]^{-1} [S_{2n}(a\tau)\, \tilde{\sigma}_{oc} + N_{2n}(a\tau)\, \tilde{\tau}_{os}]$$

(cont'd.)

$$B_{2n} = [2\mu D_n(a\tau)]^{-1} [- S_{1n}(a\tau)\, \tilde{\sigma}_{oc} - N_{1n}(a\tau)\, \tilde{\tau}_{os}] ,$$

$$(n=1,\ 2,\ \ldots) ,$$

$$A_{jo} = \frac{1}{2}[A_{jn}]_{n=o} , \qquad B_{jo} = \frac{1}{2}[B_{jn}]_{n=o} , \tag{13}$$

$$D_n(a\tau) = N_{1n}(a\tau)\, S_{2n}(a\tau) - N_{2n}(a\tau)\, S_{1n}(a\tau)$$

where:

$$\tilde{\sigma}_{oc} = \pi^{-1}\int_o^{2\pi} \bar{\sigma}_o\,(\theta,\tau)\ \cos n\,\theta\, d\theta$$

$$\tilde{\sigma}_{os} = \pi^{-1}\int_o^{2\pi} \bar{\sigma}_o\,(\theta,\tau)\ \sin n\,\theta\, d\theta$$

similarly $\tilde{\tau}_{oc}$ and $\tilde{\tau}_{os}$ are defined.

Various special cases are of interest:

(a) <u>Zero surface shear</u>: $\tilde{\tau}_{oc} = \tilde{\tau}_{os} = 0$ (14)

(b) <u>Normal traction with central symmetry and (a)</u>: $\tilde{\sigma}_{os} = 0$ (15)

(c) <u>(b) with constant amplitude over</u> $o \leq \theta \leq \alpha$:

$$(15) \text{ and } \bar{\sigma}_o\,(\theta,\tau) = \bar{\sigma}(\tau) \begin{cases} 0 \leq \theta \leq \alpha \\ \pi - \alpha \leq \theta \leq \pi \end{cases} \tag{16}$$

$$= 0 \quad \alpha < \theta \leq \pi - \alpha$$

Hence

$$\tilde{\sigma}_{oc} = 2\,\bar{\sigma}_o\,(\tau) \cdot (\pi\, n)^{-1} \sin n\alpha \ . \tag{17}$$

(d) <u>Impact load and (c)</u>:

$$(16) \text{ and } \lim_{\substack{\alpha \to 0 \\ \bar{\sigma}_o \to \infty}} 2\,\bar{\sigma}_o\,(\tau)\, a\,\alpha = \bar{P}_o\,(\tau)$$

Hence $\quad \tilde{\sigma}_{oc} = \bar{P}_o(\tau)/\pi a$ $\quad$ (18)

(e) Impulsive concentrated load and (d):

(18) and $P_o(t) = P_o \delta(t)$, where $\delta(t)$ is the Dirac delta function defined by:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

In this case,

$$\bar{P}_o(\tau) = P_o, \qquad \tilde{\sigma}_{oc} = P_o/\pi a \tag{19}$$

where P_o is the amplitude of the concentrated load.

5. Moving load

(a) Moving normal and tangential stresses:

Moving loads can be represented by

$$\sigma_o(\theta, t) = \sigma_o(\theta - \Omega t), \quad \tau_o(\theta, t) = \tau_o(\theta - \Omega_1 t) \tag{20}$$

Fourier Transforms of these are:

$$\bar{\sigma}_o(\theta, \tau) = 2 e^{i\tau\theta/\Omega} \cdot s_o(\tau/\Omega), \quad \tau_o(\theta, \tau) = 2 e^{i\tau\theta/\Omega_1} \cdot s_1(\tau/\Omega_1) \tag{21}$$

$$s_o(\tau/\Omega) = \frac{1}{2\Omega}\int_{-\infty}^{\infty} \sigma_o(\phi) e^{-i\tau\phi/\Omega} d\phi \quad \text{(in general)} \tag{22}$$

$$= \frac{1}{\Omega}\int_{0}^{\infty} \sigma_o(\phi)\cos(\tau\phi/\Omega)d\phi \quad \text{when } \sigma_o(-\phi) = \sigma_o(\phi)$$

where $s_1(\tau/\Omega_1)$ follows from $s_o(\tau/\Omega)$ by replacing Ω and $\sigma_o(\phi)$ by Ω_1 and $\tau_o(\phi)$ respectively. Hence:

$$\tilde{\sigma}_{oc} = (2\tau/\pi\Omega^2 i)\left[(\tau/\Omega)^2 - n^2\right]^{-1}(e^{2\pi\tau i/\Omega}-1)\, s_o(\tau/\Omega)$$

(23)

$$\tilde{\sigma}_{os} = (2\, n/\pi\Omega)\left[(\tau/\Omega)^2 - n^2\right]^{-1}(e^{2\pi\tau i/\Omega}-1)\, s_o(\tau/\Omega)$$

Quantities $\tilde{\tau}_{oc}$ and $\tilde{\tau}_{os}$ are obtained from (23) by writing Ω_1 and $s_1(\tau/\Omega_1)$ in place of Ω and $s_o(\tau/\Omega)$ respectively.

(b) Moving periodic loads:

$$\sigma_o(\theta - \Omega t) = (Q_o/\pi a)\sum_{n=o}^{\infty} p_n \cos n(\theta - \Omega t) + q_n \sin n(\theta - \Omega t)$$

(24)

$$\tau_o(\theta - \Omega_1 t) = (Q_o/\pi a)\sum_{n=o}^{\infty} r_n \cos n(\theta - \Omega_1 t) + s_n \sin n(\theta - \Omega_1 t)$$

After taking Fourier transforms of σ_o and τ_o, we substitute into (13) to obtain $\sigma_{oc}, \ldots, \tau_{os}$. This gives:

$$a\,\tilde{\sigma}_{oc}/Q_o = (p_n - iq_n)\,\delta(\Omega n - \tau) + (p_n + iq_n)\,\delta(-\Omega n - \tau)$$

(25)

$$a\,\sigma_{os}/Q_o = (ip_n + q_n)\,\delta(\Omega n - \tau) + (-ip_n + q_n)\,\delta(-\Omega n - \tau)$$

Quantities τ_{oc} and τ_{os} follow from (25) by writing r_n and s_n in place of p_n and q_n. In obtaining (25) we used the formal relation* [10]

$$2\pi\,\delta(u) = \int_{-\infty}^{\infty} e^{-itu}\, dt$$

Components of displacements and stress tensor can now be obtained by combining (9), (11), (13) and (25) and taking inverse Fourier transforms. Thus

---

* This is justified in the sense of a distribution function [11].

$$-(2\pi\, a\, \mu/Q_o r)\, u = \sum_{n=o}^{\infty} u_n^{(1)}(r\Omega\, n)[p_n \cos n\,(\theta-\Omega t)+q_n \sin n(\theta-\Omega t)]$$
$$+ u_n^{(2)}(r\Omega\, n)[-r_n \sin n\,(\theta-\Omega_1 t)+s_n \cos n(\theta-\Omega_1 t)]$$

$$-(2\pi\, a\, \mu/Q_o r)\, v = \sum_{n=o}^{\infty} v_n^{(1)}(r\Omega n)[-p_n \sin n(\theta-\Omega t)+q_n \cos n(\theta-\Omega t)]$$
$$+ v_n^{(2)}(r\Omega n)[-r_n \cos n(\theta-\Omega_1 t)-s_n \sin n(\theta-\Omega_1 t)]$$

$$(\pi\, a/Q_o)\sigma_{rr} = \sum_{n=o}^{\infty} \sigma_{1n}^{(1)}(r\Omega n)[p_n \cos n(\theta-\Omega t)+q_n \sin n(\theta-\Omega t)]$$
$$+ \sigma_{1n}^{(2)}(r\Omega_1 n)[-r_n \sin n(\theta-\Omega_1 t)+s_n \cos n(\theta-\Omega_1 t)] \qquad (26)$$

$$(\pi\, a/Q_o)\sigma_{r\theta} = \sum_{n=o}^{\infty} \tau_n^{(1)}(r\Omega n)[-p_n \sin n(\theta-\Omega t)+q_n \cos n(\theta-\Omega t)]$$
$$+ \tau_n^{(2)}(r\Omega n)[-r_n \cos n(\theta-\Omega_1 t) - s_n \sin n(\theta-\Omega_1 t)]$$

$$(\pi\, a/Q_o)\sigma_{\theta\theta} = \sum_{n=o}^{\infty} \sigma_{2n}^{(1)}(r\Omega n)[p_n \cos n(\theta-\Omega t)+ q_n \sin n(\theta-\Omega t)]$$
$$+ \sigma_{2n}^{(2)}(r\Omega n)[-r_n \sin n(\theta-\Omega_1 t) + s_n \cos n(\theta-\Omega_1 t)]$$

where

$$u_n^{(1)}(r\Omega n) = [D_n(a\Omega n)]^{-1} [(\alpha_1 r\Omega n)^{-1} J'_n(\alpha_1 r\Omega n) S_{2n}(a\Omega n)$$

$$- 2n(\alpha_2 r\Omega n)^{-2} J_n(\alpha_2 r\Omega n) S_{1n}(a\Omega n)]$$

$$v_n^{(1)}(r\Omega n) = [D_n(a\Omega n)]^{-1} [n(\alpha_1 r\Omega n)^{-2} J_n(\alpha_1 r\Omega n) S_{2n} a\Omega n)$$

$$- 2(\alpha_2 r\Omega n)^{-1} J'_n(\alpha_2 r\Omega n) S_{1n}(a\Omega n)] \qquad (27)$$

$$\sigma_{1n}^{(1)}(r\Omega n) = [D_n(a\Omega n)]^{-1} [N_{1n}(r\Omega n) S_{2n}(a\Omega n) - N_{2n}(r\Omega n) S_{1n}(a\Omega n)]$$

$$\tau_n^{(1)}(r\Omega n) = [D_n(a\Omega n)]^{-1} [S_{1n}(r\Omega n) S_{2n}(a\Omega n) - S_{2n}(r\Omega n) S_{1n}(a\Omega n)]$$

$$\sigma_{2n}^{(1)}(r\Omega n) = [D_n(a\Omega n)]^{-1} [T_{1n}(r\Omega n) S_{2n}(a\Omega n) - T_{2n}(r\Omega n) S_{1n}(a\Omega n)]$$

where

$$D_n(a\Omega n) = N_{1n}(a\Omega n) S_{2n}(a\Omega n) - N_{2n}(a\Omega n) S_{1n}(a\Omega n) \qquad (28)$$

Functions $u_n^{(2)}$, $v_n^{(2)}$, $\sigma_{1n}^{(2)}$, $\tau_n^{(2)}$ and $\sigma_{2n}^{(2)}$ are obtained from the corresponding ones with superscripts (1) above by replacing Ω by Ω_1 and $S_{2n}(a\Omega n)$ and $S_{1n}(a\Omega n)$ by $N_{2n}(a\Omega_1 n)$ and $N_{1n}(a\Omega_1 n)$ respectively, except in $D_n(a\Omega n)$, where we replace Ω by Ω_1.

(c) Two diametrically opposite, moving, concentrated loads (Fig. 2).

This is a case of technical importance. In this case we have $\tau_o = 0$ hence $r_n = s_n = 0$. The concentrated loads can formally be represented by

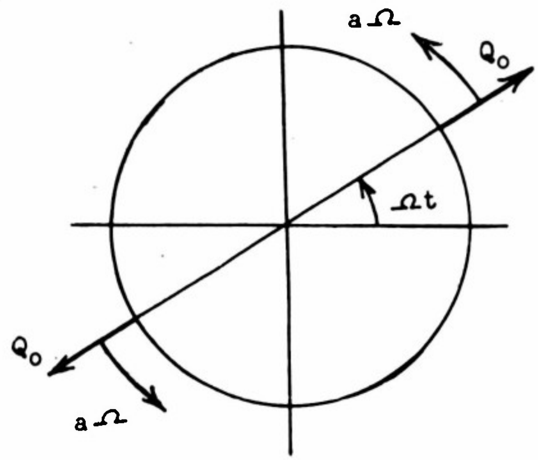

Fig. 2. Moving radial load.

$$\sigma_o(\theta-\Omega t) = (Q_o/\pi a) \left[\delta(\theta-\Omega t) + \delta(\pi - \theta + \Omega t) \right] \tag{29}$$

$$= (Q_o/\pi a) \sum_{n=o}^{\infty} p_n \cos n(\theta - \Omega t) + q_n \sin n(\theta - \Omega t)$$

where Q_o is the amplitude of each of the concentrated radial load. From (29) Fourier coefficients p_n and q_n are calculated to be:

$$p_n = \begin{cases} 2/\pi & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}, \qquad q_n = 0 \tag{30}$$

Hence (26) together with (29) and (30) and $r_n = s_n = 0$ gives the displacement and stress components. Below we give displacement components. The rest is obtained in an obvious manner.

$$-(\pi^2 a \mu/Q_o r)\, u = \sum_{0,2,4,\ldots} u_n^{(1)}(r\Omega n) \cos n (\theta - \Omega t)$$

$$(\pi^2 a \mu/Q_o r)\, v = \sum_{0,2,4,\ldots} v_n^{(1)}(r\Omega n) \sin n (\theta - \Omega t) \tag{31}$$

where $u_n^{(1)}$ and $v_n^{(1)}$ are given by (26).

Concentrated moving shear loads and other types of load combinations may easily be obtained from (24) and (26).

6. Computation and Discussion

Computations have been carried out to determine the roots of $D_n(a\Omega n) = 0$ given by (28). The roots of this equation give the resonance speed for the concentrated load.

The ratio of the resonance speed $C_r = a\Omega$ of the concentrated load to the dilatational wave velocity $C_1 = \alpha_1^{-1}$ has been solved from $D_n = 0$ for a steel cylinder with $E = 30 \times 10^6$ psi and $\nu = 0.3$. First four roots of $D_n = 0$ for $n = 1, 2, 3, 4$ are listed on the table given below*.

The computation is carried out on I.B.M. Card Program Calculator machine and are correct up to two decimal places.

The values of $n\, C_r/C_1 = n\, a\, \alpha\Omega_1$ (obtained by calculating the roots of $D_n\, (a\Omega n) = 0$). Steel: $E = 30 \times 10^6$, $\nu = 0.3$.

n	$D_1 = 0$	$D_2 = 0$	$D_3 = 0$	$D_4 = 0$
1	1.51	1.26	1.94	2.52
2	3.43	2.35	3.21	4.08
3	3.80	4.22	4.95	5.67
4	5.33	5.06	6.22	7.19

* For $n = 0$ we take the limit of the functions in (27) and find finite values.

From this table it is seen that the smallest C_r/C_1 is obtained for n=2. This value is $C_r/C_1 = 1.26/2 = .63$. We therefore expect a critical speed for the moving load in the neighborhood of 0.63 C_1 which will create resonance in the disk.

The Rayleigh surface wave velocity for $\nu = .29$ is 0.9258 times the velocity of shear waves or .503 times the velocity of dilational waves. Thus 0.63 C_1 represents a velocity between the Rayleigh surface wave velocity and the shear wave velocity. Further we notice that this velocity is minimum for n=2 rather than n=1. Examining (31) we found that we have only even terms. Hence diametrically opposite moving loads give a smaller critical speed than the case of a single load. In the former case the fundamental mode n=0 represents a uniform lateral extension which is not dependent on time. Thus it is static in nature and has no resonance frequency associated with it. The second mode is of cos 2 θ —type and gives the minimum critical speed mentioned above. It is interesting to know that the minimum critical speed in the case of a flat semi-infinite plate (the Rayleigh surface wave velocity) is less than that of the curved surface (the disk in the present case). With this point in mind perhaps one can classify the curved surfaces from the elasto-dynamic point of view. This process will, no doubt, present mathematical difficulties.

The present results, of course, must be accepted tentatively until experimental verification.